A note on a relation between ac Josephson effect and double-well BEC oscillations

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(Dated: December 17, 2009)

In this brief note we comment on the relation between the ac Josephson effect and the coherent oscillations of a Bose-Einstein condensate confined to a double-well potential. The goal is to elucidate the extent to which the latter is a realization of the former. We detail the correspondence that emerges in the high occupation limit of the double-well potential, and particularly note the relation between the two oscillation frequencies.

A realization of trapped degenerate atomic gases has opened opportunities to study many interesting quantum many-body phenomena in previously unexplored regimes[1]. Recent experiments[2, 3] on Bose-Einstein condensates (BEC), trapped and oscillating in an imbalanced double-well potential have sought to realize an atomic Bose gas analog[4, 5, 6, 7, 8] of the alternating- and direct-current (ac and dc) Josephson effects[9, 10].

While there are some obvious analogies, considerable fundamental differences between the two systems exist. For instance, a conventional Josephson junction (JJ) between two superconductors is an open quantum many-body system driven by a fixed electro-chemical potential (voltage) difference between the left and right contacts, or with a current imposed by an electrical circuit. Thus on general grounds (gauge invariance and Heisenberg equation of motion) the evolution of the relative phase $\phi = \phi_L - \phi_R$ in an ac Josephson effect is given, *exactly*, by

$$\hbar\dot{\phi} \;=\; 2eV, \tag{1}$$

where $eV = \mu_L - \mu_R$ is the imposed electro-chemical potential difference across the left and right contacts controlled by the voltage V. When combined with the (lowest harmonic) expression for the Josephson current, $I = I_0 \sin\phi$, the above exact *linear* growth of $\phi(t)$ with time gives the standard ac Josephson effect,

$$I(t) = I_0 \sin(\omega_J t),\ \omega_J = 2eV/\hbar. \tag{2}$$

In addition to its basic physical importance it provides an extremely accurate voltage-frequency relation (thereby defining a Volt to one part in 10^8) with a number of other important applications[10]. In a conventional superconducting JJ the critical current is given by the standard Ambegaokar-Baratoff formula[11], $I_0 = \pi\Delta G_n/(2e)$, where Δ is the superconducting (pairing) gap and G_n is the normal state Josephson junction conductance.

In contrast, a double-well BEC is a closed system with only the total number of atoms fixed and is not necessarily in the thermodynamic limit. Deep in the condensed state and for deep wells the coherent dynamics can be studied via two coupled (Gross-Petaevskii) equations of motion for two coherent-state amplitudes $\Psi_{L,R}(t)$, with $|\Psi_{L,R}|^2 = N_{L,R}(t)$ giving the number of atoms in the left and right wells, with only the total number $N = N_L + N_R$ of atoms conserved. The dynamics is given by Euler-Lagrange equations for the coherent-state action given by

$$S = \int dt \left[\Psi_L^* i\hbar\partial_t \Psi_L + \Psi_R^* i\hbar\partial_t \Psi_R - H(\Psi_L, \Psi_R)\right], \tag{3}$$

with the Hamiltonian

$$\begin{aligned} H \;=\; & -J\Psi_L^*\Psi_R - J\Psi_R^*\Psi_L + \varepsilon_L|\Psi_L|^2 + \varepsilon_R|\Psi_R|^2 \\ & + \frac{g_0}{2N}|\Psi_L|^4 + \frac{g_0}{2N}|\Psi_R|^4 + \frac{g_1}{N}|\Psi_L|^2|\Psi_R|^2, \end{aligned} \tag{4}$$

and parameters $J, \varepsilon_{L,R}, g_i$ straightforwardly derivable from a continuum model of interacting bosons trapped in a double-well potential[5]. Since the Hamiltonian should scale linearly with the system size, for convenience we defined the interaction couplings with explicit factors of $1/N$ so that the parameters g_i do not scale with the system size.

For $g_i = 0$ the dynamics reduces to that of a two-level system (e.g., spin in a magnetic field), with oscillations (that we will loosely call Rabi oscillations) arising from non-eigenstate initial conditions. This two level system has a matrix representation

$$\mathcal{H} = \begin{pmatrix} \varepsilon_L & -J \\ -J & \varepsilon_R \end{pmatrix} \tag{5}$$

with well-known eigenvalues

$$E_\pm = \varepsilon \pm \sqrt{J^2 + h^2}, \tag{6}$$

where we defined

$$\varepsilon = \frac{1}{2}(\varepsilon_L + \varepsilon_R),\ h = \frac{1}{2}(\varepsilon_L - \varepsilon_R) \tag{7}$$

(with the energy difference h not to be confused with the Planck constant, for which we will use the symbol $2\pi\hbar$ throughout). Thus any quantity quadratic in Ψ_L and Ψ_R will oscillate with the Rabi frequency

$$\omega_R = (E_+ - E_-)/\hbar = 2\sqrt{J^2 + h^2}/\hbar. \tag{8}$$

Although this double-well BEC system was studied in a seminal work by Smerzi, et al.[5] and a number of works that followed, from the recent discussion[12] of the latest experiment[3] it appears that a number of questions

remain unanswered, namely: (i) How is the Rabi frequency, ω_R, which depends on system-specific quantities such as J, related to the universal Josephson frequency, ω_J in (2), which depends only on the applied voltage? (ii) Moreover, what role do interactions (clearly neglected in ω_R) and the thermodynamic limit play in establishing the relation between the two systems and corresponding frequencies?

A connection between these (otherwise quite distinct) systems only exists in the specific limit of macroscopic wells, $L_{L,R} \to \infty$ (reached for large occupation $N_{L,R} \gg 1$) and a finite barrier thickness $d \ll L$ of the double-well BEC system. This is necessary in order to approximately model the thermodynamically large lead reservoirs of a superconducting Josephson junction. Our key observation is that in this limit the Josephson coupling, J, as defined by (4) vanishes as $1/L$, i.e., vanishes in the large atom occupation number limit, and thus ω_R goes over to ω_J, exactly in the thermodynamic limit, with the identification of h with eV. Moreover, in this limit the amplitude of oscillations in the number imbalance is always small regardless of the interaction strength, and thus the nonlinearity plays negligible role in their dynamics, aside from just redefining the relationship between h and V to be

$$eV \leftrightarrow h + (g_0 - g_1)\,\frac{N_L^{(0)} - N_R^{(0)}}{2N}, \tag{9}$$

with $N_{L,R}^{(0)}$ the equilibrium number of atom in the two wells.

The vanishing of J for macroscopic leads and fixed barrier width d can be understood on general grounds simply by noting that J is an interfacial energy per particle, associated with the coupling of the left and right leads. It is therefore proportional to the surface area of the barrier $A \sim L^2$ divided by system's volume, i.e., $J \sim L^2\ell/L^3 \sim \ell/L$, (with ℓ the inverse penetration length scale set by the barrier width and height) and thus indeed vanishes in the above thermodynamic limit.

To derive these results more explicitly, we estimate the typical size of J in a macroscopic system by solving the Schrödinger equation of a particle of mass m_a in a double-well symmetric potential, $U(z) = U(-z)$, with the difference between the first excited and ground states by definition giving $2J$. This problem is set up in Ref. 13 and consists of constructing wave functions $\psi_0(z)$ and $\psi_0(-z)$ localized in the left and right wells, respectively. It is then shown in a straightforward way that the energy splitting in this double well is given by

$$J = -\frac{\hbar^2}{m_a}\psi_0(0)\psi_0'(0). \tag{10}$$

To evaluate this expression, it is crucial to distinguish cases of a smooth and sharp barriers. The case of a smooth barrier is solved in Ref. 13 using the WKB approximation and leads to the expression for J proportional to the natural frequency of the oscillations (attempt frequency) in each of the wells as well as the dimensionless coefficient of penetration through the barrier. In contrast, motivated by the connection to the JJ problem we are instead interested in a sharp barrier (relative to the size L of the wells) of width d, located between points $z = -d/2$ and $z = d/2$ (z is the axis along the leads and perpendicular to the barrier), where the potential energy exceeds the kinetic energy by the amount U_0. For this setup the WKB is clearly inapplicable. Under such conditions, the wave function in the wells is approximately

$$\psi_0(z) \approx \frac{1}{\sqrt{L}}e^{ikz},\ z < -\frac{d}{2}, \tag{11}$$

where k is the wave vector in the wells, while the wave function under the barrier is

$$\psi_0(z) \approx \frac{1}{\sqrt{L}}e^{-(z+d/2)/\ell},\ -\frac{d}{2} \le z \le 0, \tag{12}$$

with length $\ell = \hbar/\sqrt{2m_aU_0}$. This then gives[14]

$$J = \frac{\hbar^2}{m_a\ell L}e^{-d/\ell}. \tag{13}$$

Another more direct way of producing Eq. (13) is to note that J is proportional to the matrix element of the Hamiltonian (e.g., the kinetic energy) between the left and right wavefunctions, normalized in each of the wells and penetrating a distance ℓ into the barrier. Clearly then $J \approx \int_{-L}^{L} dz \frac{\hbar^2}{2m_a}\psi_L^*\psi_R''$ giving result (13), and as advertized vanishes for macroscopically large wells.

Consequently, for any realistic energy difference $2h$ between the two wells, J in (8) can be neglected, reducing the Rabi oscillation frequency for such large wells to

$$\omega_R = 2h/\hbar, \tag{14}$$

and allowing the identification of the imbalance h in the double-well BEC with the chemical potential difference (voltage) eV, (2).

We now turn to the analysis of the effects of interactions on our conclusion above by considering the Hamiltonian (4) with $g_i \neq 0$. This is most conveniently done in the (polar) density-phase representation, by introducing

$$\Psi_L = \sqrt{N_L}e^{i\phi_L t},\ \Psi_R = \sqrt{N_R}e^{i\phi_R t}, \tag{15}$$

and

$$\begin{aligned} N_L &= \frac{N}{2}(1+m), \quad \phi_L = \theta + \phi, \\ N_R &= \frac{N}{2}(1-m), \quad \phi_R = \theta - \phi, \end{aligned} \tag{16}$$

where $-1 \le m \le 1$. The total number of particles N is conserved (hence H is independent of θ), and the part of the Hamiltonian describing the evolution of m becomes

$$\mathcal{H} = \frac{H}{N} = -J\sqrt{1-m^2}\cos(2\phi) + hm + \frac{\lambda}{2}m^2, \tag{17}$$

where $\lambda = (g_0 - g_1)/2$. The equations of motion are thus

$$\begin{aligned} \hbar\dot{\phi} &= -\frac{\partial \mathcal{H}}{\partial m}, \\ &= -h - \lambda m - J\frac{m}{\sqrt{1-m^2}}\cos(2\phi) \equiv \frac{\delta\mu}{2}, \qquad (18) \\ \hbar\dot{m} &= \frac{\partial \mathcal{H}}{\partial \phi} = 2J\sqrt{1-m^2}\sin(2\phi), \qquad (19) \end{aligned}$$

with (18) defining the effective chemical potential difference $\delta\mu = \mu_L - \mu_R$. Although it is not fixed in this canonical ensemble, in the thermodynamic limit of interest its constant (in time) part can be meaningfully associated with the grand-canonical chemical potential different of the JJ system. Above equations provide a complete solution to the problem of two couple interacting condensates and have been analyzed in Ref. 5. Although it is possible to solve these equations analytically, the general solution is not very informative.

Here we focus on the physical limit of large reservoirs discussed above, in which the coupling J is vanishingly small. In this limit we solve Eqs. (18) perturbatively in powers of $J/h \ll 1$. At zeroth order in J/h,

$$\begin{aligned} \hbar\dot{\phi}_0 &= -h - \lambda m_0, \qquad (20) \\ \dot{m}_0 &= 0, \qquad (21) \end{aligned}$$

leading to the solution

$$\begin{aligned} \phi_0(t) &= -\frac{h + \lambda m_0}{\hbar} t \equiv -\tilde{h}t, \qquad (22) \\ m_0 &= \text{const.} \qquad (23) \end{aligned}$$

Identifying atom current I with $(\dot{N}_R - \dot{N}_L)/2 = -N\dot{m}/2$, we obtain, with the help of Eq. (19),

$$I = \frac{JN\sqrt{1-m_0^2}}{\hbar}\sin\left(\frac{2\,(h+\lambda m_0)\,t}{\hbar}\right), \qquad (24)$$

that is clearly equivalent to (2), with the identification (9) and $I_0 = JN\sqrt{1-m_0^2}/\hbar$.

From this last identification of I_0 we can further note that the result Eq. (13) is compatible with the Ambegaokar-Baratoff expression for the critical current in a Josephson junction, namely, that

$$I_0 \sim \frac{N}{L\ell} = \frac{\rho L^2}{\ell}, \qquad (25)$$

with $\rho = N/L^3$ the atom density. From this we observe that the critical current of the double-well BEC scales as the area L^2 of the "junction" (double-well barrier), just like the Ambegaokar-Baratoff expression for the superconducting JJ, $I_0^{AB} = \pi\Delta G_n/(2e)$, where the junction area enters through the number of conduction channels in G_n. This further supports our finding that J, as defined in (4) vanishes with the inverse length of the well "leads".

Let us see whether any realistic Josephson junctions indeed obey the condition $J \ll h$. For a typical Josephson junction, $d \sim 100$nm [15]. ℓ must itself be of the order of d, otherwise the coupling J will be even further exponentially suppressed. Taking $L \gtrsim 100$nm and $h \sim 1$eV, we find

$$\frac{J}{h} = \frac{\hbar^2}{m_a \ell L h} \lesssim 10^{-5}, \qquad (26)$$

where an electron mass was used for m_a. Roughly, the above factor of 10^{-5} arises due to two factors of 10^3 of ℓ and L (large characteristic length scales) relative to the Bohr radius that corresponds to an eV energy scale. Thus indeed in a realistic Josephson junction J is always much smaller than h, well justifying above approximation, even if the system were to be closed. Of course, as mentioned in the introduction, a conventional superconducting Josephson junction is an open system, that is a part of a macroscopic circuit and is therefore effectively characterized by an infinite L. Furthermore, with the circuit driven by a fixed voltage source the Josephson frequency expression ω_J, (2) is effectively exact; any inaccuracy in Josephson voltage-frequency relation quoted above is associated with the uncertainty of the current knowledge of Planck's constant (one part in 10^7)[16].

In contrast a typical double-well BEC trap potential is expected to be roughly characterized by a single (the same order of magnitude) length and energy scale, with $\ell \sim L$ and $U_0 \sim h$, leading to J and h, that are comparable and both a tiny fraction of an electron volt. Consequently, we expect such a system to display a significant and tunable deviation from the Josephson frequency, Eq. (22), obtained by neglecting $J/\tilde{h}$ corrections.

Even though as discussed above in a double-well BEC system we generically expect $J \sim h$, we observe that the experimentally studied double-well BEC[3] is characterized by $J/\hbar \approx 15$ sec^{-1}, and $1400 \lesssim h/\hbar \lesssim 5700$ sec^{-1} (here and throughout h denotes the chemical potential imbalance energy, not the Planck's constant $2\pi\hbar$), and thus corresponds to $0.003 \lesssim J/h \lesssim 0.011$.[17] This small value of J/h characterizing these experiments explains why the measured oscillation frequency (ω_{acJ}) as a function of the chemical potential difference (h) is observed to be linear in Ref. 3. In order to detect a deviation from this linear behavior these measurements need to be extended down to $h/\hbar \lesssim 15$ sec^{-1}, or done on a system in which the value of J is increased by e.g., making wells smaller.

With this in mind, it is useful to compute the lowest order correction. This can be straightforwardly done by evaluating the solutions $\phi(t), m(t)$ to Eq. (18) systematically to nth order in J/h by iterating the equations, with the nth-order solution $\phi_n(t), m_n(t)$ on the left-hand side and approximating the right-hand side by the $n-1$st-order solution, $\phi_{n-1}(t), m_{n-1}(t)$.

To first-order the equations become:

$$\dot{\phi}_1 = -\tilde{h} - J\frac{m_0}{\sqrt{1-m_0^2}}\cos(2\phi_0), \quad (27)$$

$$\dot{m}_1 = 2J\sqrt{1-m_0^2}\sin(2\phi_0), \quad (28)$$

leading to the solution

$$\phi_1(t) = -\tilde{h}t - \frac{J}{2\tilde{h}}\frac{m_0}{\sqrt{1-m_0^2}}\sin(2\tilde{h}t), \quad (29)$$

$$m_1(t) = m_0 + \frac{J}{\tilde{h}}\sqrt{1-m_0^2}\cos(2\tilde{h}t). \quad (30)$$

The solution to second-order, $(J/\tilde{h})^2$ is obtained by using above expressions, $\phi_1(t), m_1(t)$ in the right-hand side of the exact equations (18):

$$\dot{\phi}_2 = -\tilde{h} - J\frac{m_1}{\sqrt{1-m_1^2}}\cos(2\phi_1), \quad (31)$$

$$\dot{m}_2 = 2J\sqrt{1-m_1^2}\sin(2\phi_1). \quad (32)$$

Focusing on $\phi(t)$ and integrating the first equation we find the time-independent part of $\dot{\phi}$

$$\dot{\phi}_2 = -\tilde{h} - J\frac{m_0+\delta m(t)}{\sqrt{1-(m_0+\delta m(t))^2}}\cos\left[2\tilde{h}t + \frac{J}{\tilde{h}}\frac{m_0}{\sqrt{1-m_0^2}}\sin(2\tilde{h}t)\right], \quad (33)$$

$$\approx -\tilde{h} - \frac{Jm_0}{\sqrt{1-m_0^2}}\cos(2\tilde{h}t) + \frac{J^2}{\tilde{h}}\frac{m_0^2}{1-m_0^2}\sin^2(2\tilde{h}t) - \frac{J^2}{\tilde{h}}\frac{1}{1-m_0^2}\cos^2(2\tilde{h}t), \quad (34)$$

$$\approx -\tilde{h} + \frac{J^2}{2\tilde{h}}\frac{m_0^2}{1-m_0^2} - \frac{J^2}{2\tilde{h}}\frac{n^2}{1-m_0^2} + \ldots, \quad (35)$$

$$\approx -\tilde{h} - \frac{J^2}{2\tilde{h}} \approx -\sqrt{\tilde{h}^2+J^2}, \quad (36)$$

which by definition (18) is the effective chemical potential difference. In above we have neglected the higher order hamonics that are also always generated at nonzero $J/\tilde{h}$. This then gives

$$\phi(t) \approx -t\sqrt{\tilde{h}^2+J^2}, \quad (37)$$

$$\dot{m} \approx -2J\sqrt{1-m_0^2}\sin\left[2t\sqrt{\tilde{h}^2+J^2}\right]. \quad (38)$$

We therefore obtain the (fundamental) ac Josephson frequency of current oscillations (defined by $\dot{m}$) to be given by

$$\omega_{acJ} \approx \tilde{h} + \frac{J^2}{2\tilde{h}} \approx \sqrt{\tilde{h}^2+J^2}, \quad (39)$$

with $\tilde{h} = h + \lambda m_0$.

Thus, as advertized, for macroscopically occupied wells (the only limit in which a reasonable connection to a Josephson junction can be made), such that $J/\tilde{h} \ll 1$, an imbalanced double-well BEC system is indeed a good model for the ac Josephson effect, exhibiting current oscillations with frequency that is nearly independent of J and grows linearly with imbalance $\tilde{h} = h + \lambda m_0$. However, for a smaller double-well BEC (more typical experimentally) the fundamental oscillation frequency is expected to exhibit $(J/\tilde{h})^2$ deviations from the linear dependence on $\tilde{h}$ of the form given in (39).

We thank S. Benz, K. Lehnert, W. Phillips, and J. Steinhauer for discussions, appreciate the hospitality of the Kavli Institute for Theoretical Physics in China during the "Condensed Matter Physics of Cold Atoms" workshop, when part of this work was performed, and acknowledge financial support by the NSF grants DMR-0321848 (L.R.), DMR-0449521 (V.G.), and PHY-0904017 (V.G.).

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